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A convergence theorem for asymptotic expansions of Feynman amplitudes

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Abstract. The Mellin representation of Feynman integrals is revisited. From this representation, an asymptotic expansion for generic Feynman amplitudes, for any set of invariants going to zero or to ∞, may be obtained. In the case of all masses going to zero in a Euclidean metric, we show that the truncated expansion has a remainder which is compatible with convergence of the series.

1. Introduction

Divergences in field theories have been historically considered as an undesirable feature, a kind of 'illness' of the theory which should be 'cured' at any price. The most striking example is ultraviolet (UV) divergences, coming from the non-definiteness of field products at the same spacetime point, and its 'cure', the various renormalization procedures. Also, the divergent large-distance behaviour of theories containing massless fields, infrared divergences, received an equivalent amount of attention over the last few decades. Infrared divergences may be seen as a special case of a general class of asymptotic behaviours of Feynman amplitudes in a field theory (which includes also UV divergences), as some of the masses involved tend to zero. Actually, these divergences appear at different levels. For Green functions in a Minkowskian metric it was shown a long time ago that for some theories (e.g. QED) Green functions exist at the zero-mass limit for some particles, as distributions on the 4-momenta, i.e. Green functions are well defined quantities in the infrared limit [1]. For particles on-mass-shell Green functions generally do not have a limit for these theories, even if they are well defined off-mass-shell Green functions. The oldest and best known example is that of infrared divergences in scattering amplitudes in QED. This problem has been investigated exhaustively (classical papers on the subject are in [3, 4]), since the work of Bloch and Nordsieck [5].

Another class of problems arise at the Green functions level in the Euclidean metric, when besides the zero-mass limit, vanishingly small values for the external momenta are also considered. In this case, we speak of the infrared behaviour of correlation functions. These divergences, which are seen as a 'pathological' behaviour in the context of the applications of field theories to particle physics, are associated with the large-distance correlations in statistical systems and play a crucial role in the study of critical phenomena and phase transitions. Since the Ginzburg–Landau model was introduced, half a century ago [6], as a phenomenological model for superfluidity and superconductivity, the idea that field theory models can describe statistical systems near criticality became well established. Indeed, in its one-component version, the Ginzburg–Landau model has been used with remarkable success as a continuous statistical mechanics model for the critical phenomena of systems lying in the same universality class as the Ising model. In its N-component version coupled to Abelian gauge fields it has been used as a model for superconductivity and liquid crystals [7]. Presently, the basic field-theoretical approach to study critical phenomena is the renormalization group analysis of flows in the neighbourhood of fixed points. In particular, recently some works have used this technique to study the topological Ginzburg–Landau model. For instance, in [8] a Ginzburg–Landau model for superconductivity with a Chern–Simons term added is considered, and a similar study has also been done in [9]. On more general grounds important works on the subject are in [10–13]. Another approach, using the finite-temperature field-theory formalism has also been used recently (see, for instance [14, 15].

It is worthwhile emphasizing that, in contrast to what happens in applications to particle physics, in applications of field theory to critical phenomena both ultraviolet divergences and infrared behaviour need not be 'cured'. The ultraviolet cut-off is related to the inverse of some fundamental length of the system such as the atomic scale and the infrared behaviour of correlation functions describes directly the approach to critical points.

In this paper, we adopt a mathematical physicist's point of view and consider both ultraviolet divergences and infrared behaviour as particular cases of the asymptotic behaviour of Feynman amplitudes in the Euclidean metric. As we clarify below, a Feynman amplitude depends on a set of invariants built from external momenta and on squared masses and by asymptotic behaviour we mean the behaviour of such an amplitude as some of these invariants and or squared masses tend to zero or to infinity. We make use of Mellin transform techniques to represent Feynman integrals, along similar lines to that used to study renormalization and asymptotic behaviour of scattering amplitudes in [16–18], and to study the heat kernel expansion as in [19]. To fix our framework we consider a theory involving scalar fields $\varphi_i(x)$ having masses m_i, defined on a Euclidean space. For simplicity we may think of a single scalar field $\varphi(x)$ having a mass m. A generic Feynman graph G is a set of I internal lines, L loops, q connected components (a graph is disconnected if $q > 1$) and n vertices linked by some (polynomial) potential. To each vertex are attributed external momenta $\{p_i\}$ and internal ones $\{k_a\}$. A subgraph $S \subset G$ is a graph such that all the lines vertices and loops belong to G and a quotient graph $\frac{G}{S}$ is a graph obtained from G reducing S to a point. A q-tree of the diagram G is a subgraph of G having q connected components, without loops and linking all the vertices of G. Of particular interest for us are the cases $q = 1$ (1-trees) and $q = 2$ (2-trees).

The Feynman amplitude $G(\{a_k\})$ corresponding to G is a function of the set of invariants $\{a_k\}$ built from external momenta $\sum p^2$ and squared masses m_i^2; it is defined in the Schwinger–Bogoliubov representation by (see for instance [1, 2])

$$G(a_k) = \int_0^\infty \prod_{i=1}^{I} \mathrm{d}\alpha_i \, U^{-D/2}(\alpha) \, \mathrm{e}^{-V(\alpha)/U(\alpha)} \tag{1}$$

where D is the space dimension with a positive metric.

In the above formula, the Symanzik polynomials $U(\alpha)$ and $V(\alpha)$ are constructed from the graph G by the prescription

$$U(\alpha) = \sum_{1.T} \prod_{i \notin 1.T} \alpha_i \tag{2}$$

and

$$V(\alpha) = \sum_{2.T} \left(\sum p_j \right)^2 \left(\prod_{i \notin 2.T} \alpha_i \right) + \left(\sum_{j \in G} m_j^2 \alpha_j \right) U(\alpha) \tag{3}$$

where the symbols $\sum_{1.T}$ and $\sum_{2.T}$ mean, respectively, summation over the 1-trees and 2-trees of G. The sum $\sum p_j$ in equation (3) is the total external momentum entering one of the 2-tree connected components (any one of them equivalently, by momentum conservation). Note that $U(\alpha)$ and $V(\alpha)$ are homogeneous polynomials in the α-variables, of degrees L and $L+1$, respectively.

2. The Mellin representation of Feynman integrals and asymptotic expansions

In the following we have in mind as a physical situation, the infrared behaviour, but we would like to emphasize that our study is quite general, in the sense that it applies to any asymptotic limit in the Euclidean metric (any choice of the subset a_l below), for arbitrarily given external momenta, generic or exceptional, and for arbitrary vanishing or finite masses. If we perform a scale transformation on the subset $\{a_l\}$ of invariants, $a_l \to \lambda a_l$, the polynomial V splits into two parts

$$V(\lambda a_m) = \lambda W(a_l, \alpha) + R(a_q, \alpha) \tag{4}$$

where the polynomials $W(a_l, \alpha)$ and $R(a_q, \alpha)$ are also homogeneous of degree $L+1$ in the α-variables.

To be concrete we consider here a special situation with the external momenta $\{p\}$ fixed and we investigate the limit $\lambda \to 0$ corresponding to vanishing masses. In this case W is just the second term in equation (3). As we have noted above, the method applies along the same lines to any other class of asymptotic behaviour. We note that from a dimensional argument

$$G\left(\frac{a_l}{\lambda}, a_q\right) = \lambda^{\omega} G(a_l, \lambda a_q) \tag{5}$$

where $\omega = I - \frac{1}{2}DL$, the study of a given subset going to zero is equivalent to studying the $\lambda \to \infty$ limit on the complementary subset of invariants.

Under the λ-scaling performed in equation (4) G becomes a function of λ, $G(\lambda)$, and its Mellin transform, $M(z) = \int_0^\infty \mathrm{d}\lambda\, \lambda^{-z-1} F(\lambda)$ may be written in the form

$$M(z) = \Gamma(-z) \int_0^\infty \prod_{i=1}^{I} \mathrm{d}\alpha_i\, U^{-D/2} \mathrm{e}^{-R/U} \left(\frac{W}{U}\right)^z \tag{6}$$

where, as can be seen from equations (2)–(4), $\frac{W}{U}$ is a real positive polynomial in the α-variables and where $\sigma = \mathrm{Re}(z) < 0$ belongs to the analyticity domain of $M(z)$.

The scaled amplitude associated with the Feynman graph G, $G(\lambda)$, may be obtained by the inverse Mellin transform

$$G(\lambda) = \frac{1}{2\mathrm{i}\pi} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathrm{d}z\, \lambda^z M(z). \tag{7}$$

Since the integrand of equation (7) vanishes exponentially at $\sigma \pm \mathrm{i}\infty$ due to the behaviour of $\Gamma(z)$ at large values of Im z (see, for instance, [22]), the integration contour may be displaced to the right by Cauchy's theorem, picking up successively the poles of the integrand, provided we can desingularize the integral in equation (6). Such a problem has been studied by an appropriate choice of local coordinates in [20] and also in [16] using Hepp sectors and a multiple Mellin representation. In these works it has been possible to show that the meromorphic structure of $M(z)$ has the form

$$M(z) = \sum_{n,q} \frac{A_{nq}\, q!}{(z-n)^{q+1}}. \tag{8}$$

It results from the displacement of the integration contour in the inverse Mellin transform, an expansion for small values of λ, of the form

$$G(\lambda) = \sum_{n=n_0}^{N} \lambda^n \sum_{q=0}^{q_{\max}(n)} A_{nq} \ln^q(\lambda) + R_N(\lambda) \tag{9}$$

where the coefficients $A_n(\{p\})$ and the powers of logarithms come from the residues at the poles $z = n$. Generally, for each n in equation (9) $q_{\max}(n)$ is always smaller than n, in such a way that for small λ the large-n behaviour and the convergence rate of the expansion are controlled by the powers of λ.

The remainder of the expansion, $R_N(\lambda)$, is given by

$$R_N(\lambda) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}(\mathrm{Im}\, z)}{2\mathrm{i}\pi} \lambda^z \Gamma(-z) F(z) \tag{10}$$

with

$$\mathrm{Re}(z) = N + \eta \qquad 0 < \eta < 1 \tag{11}$$

and where

$$F(z) = \int_0^{\infty} \prod_{i=1}^{I} \mathrm{d}\alpha_i\, U^{-D/2} \mathrm{e}^{-R/U} \left(\frac{W}{U}\right)^z. \tag{12}$$

To perform the α-integrations explicitly in equation (6) above for a general Feynman amplitude is a rather difficult task. As this calculation will not be necessary for our purposes in this work, in the remainder of this section we sketch the general lines of how it can be done, for convergent as well as for divergent diagrams and give the appropriate references for the interested reader. As a matter of technique, we can divide the α-domain of integration into $I!$ Hepp sectors by ordering the α's in all possible ways. To a given permutation $h = (i_1, i_2, \ldots, i_I)$ there corresponds a sector

$$\alpha_{i_1} \leqslant \cdots \leqslant \alpha_{i_I} \tag{13}$$

with corresponding sector variables

$$\alpha_{i_I} = \beta_I \qquad \alpha_{i_{I-1}} = \beta_I \beta_{I-1} \qquad \ldots \qquad \alpha_{i_1} = \beta_I \beta_{I-1} \ldots \beta_2 \beta_1. \tag{14}$$

Using the homogeneous properties in the α-variables of the polynomials R, U and W in the integrand of $M(z)$, the integration over β_I can be made explicitly and the remaining integration domain is $0 \leqslant \beta_i \leqslant 1, i = 1, \ldots, I-1$. In Euclidean metric it is shown in [16, 17] that the Symanzik polynomials are factorized in each Hepp sector, which implies that in each sector the α ordering induces for the integrand of the Mellin transform in equation (6), $P(\{\alpha\}; z)$, a factorization of the form

$$P(\{\alpha\}; z) \to \left(\prod_i \beta^{r_i}\right) Q(\{\beta\}; z) \tag{15}$$

such that $Q(\{0)\}; z \neq 0$. The behaviour of $P(\{\alpha\}; z)$ around zero is governed by its vanishing when subsets of α-variables go to zero linearly. Convergence or divergence of the integral $\int_0^\infty \prod_i \mathrm{d}\alpha_i\, P(\{\alpha_i\}; z)$ can then be determined by power counting. For the same reason, eventual divergences in such an integral can be removed by Taylor subtractions [23, 24]. It is also shown in [16, 17] that the case of diagrams for which zeros of $U(\alpha)$ induce ultraviolet divergences can be treated as in the convergent case along the following lines: in the α-parametric representation, these divergences are renormalized by Taylor subtractions. Then it can be shown using an integral formula for the remainder of Taylor expansions, that the renormalized Feynman amplitude is a finite sum of convergent integrals which are exactly of the same type as in the case of convergent diagrams, provided the various integration variables associated with the remainders of the Taylor series are renamed as supplementary Hepp β-variables. In the following we keep the notation corresponding to convergent graphs, understanding that the results are valid for convergent as well as for renormalized divergent diagrams.

3. The remainder of the expansion

Thus we have a first bound to the remainder R_N in the truncated expansion above:

$$|R_N(\lambda)| \leqslant \lambda^{N+\eta} Q_N \tag{16}$$

with

$$Q_N = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\beta}{2\pi} |\Gamma(-N-\eta-\mathrm{i}\beta) F(\sigma+\mathrm{i}\beta)|. \tag{17}$$

Using recurrence formulae we may relate $\Gamma(-\sigma-\mathrm{i}\beta)$ to a gamma function which has positive real part of the argument. We obtain, remembering equation (11) above,

$$\Gamma(-N-\eta-\mathrm{i}\beta) = \Gamma(2-\eta-\mathrm{i}\beta) \prod_{j=0}^{N+1} \frac{1}{(-N-\eta+j)-\mathrm{i}\beta}. \tag{18}$$

Now, it may be shown [21] that for $c > 0$ the gamma function $\Gamma(c-\mathrm{i}\beta)$ is bounded in absolute value

$$|\Gamma(c-\mathrm{i}\beta)| \leqslant \mathrm{e}^{-\epsilon|\beta|} \int_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{e}^{cu-\mathrm{e}^{u}\cos\epsilon} \tag{19}$$

where $\epsilon < \frac{1}{2}\pi$ is a strictly positive constant independent of c and β and $c = 2-\eta$ is also a positive constant. Thus the integral in the equation above is convergent, implying that the bound has the form

$$|\Gamma(c-\mathrm{i}\beta)| < c' \mathrm{e}^{-\epsilon|\beta|}. \tag{20}$$

From equations (17), (18) and (20) we have

$$Q_N < c' \int_{-\infty}^{\infty} \frac{\mathrm{d}\beta}{2\pi} |F(N+\eta+\mathrm{i}\beta))| \prod_{j=0}^{N+1} \frac{1}{[(-N-\eta+j)^2+\beta^2]^{\frac{1}{2}}}. \tag{21}$$

Also we have the inequalities

$$\prod_{j=0}^{N+1} \frac{1}{[(-N-\eta+j)^2+\beta^2]^{\frac{1}{2}}} \leqslant \prod_{j=0}^{N+1} \frac{1}{|(-N-\eta+j)|} < \frac{1}{N!\eta|(1-\eta)|}. \tag{22}$$

The first one is obvious, since $\beta^2 \geqslant 0$. To see the second one, let us adopt for a moment the notation $\sigma = N+\eta$, and write

$$\prod_{j=0}^{N+1} \frac{1}{|-\sigma+j|} = \prod_{j=0}^{N+1} \frac{1}{|\sigma-j|} = \frac{1}{\sigma(\sigma-1)\ldots(\eta+1)\eta|\eta-1|}. \tag{23}$$

From $\sigma > N, \sigma - 1 > N - 1, \ldots, \eta + 1 > 1$, we find

$$\prod_{j=0}^{N+1} \frac{1}{|-\sigma + j|} < \frac{1}{N!\eta(1-\eta)}. \tag{24}$$

Combining equations (22), (21) and (16) we obtain a bound for the remainder of the truncated expansion for the λ-scaled amplitude $G(\lambda)$:

$$|R_N(\lambda)| < \frac{\lambda^N c'}{N!\eta|(\eta - 1)|} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\beta}{2\pi} \mathrm{e}^{-\epsilon|\beta|} |F(N + \eta + \mathrm{i}\beta)|. \tag{25}$$

Displacing indefinitely the integration path would generate instead of the truncated expansion, a series, provided the remainder R_N had an appropriate behaviour as $N \to \infty$. Let us particularize to the limit of all masses going to zero. In this case the function $F(z)$ in equation (12) has the form

$$F(z) = (\mu^2)^z \int_0^\infty \prod_{i=1}^{I} \mathrm{d}\alpha_i \, U^{-D/2} \mathrm{e}^{-R/U} \left(\frac{W'}{U}\right)^z \tag{26}$$

where μ is a constant mass parameter and $W' = (\sum_{j \in G} \alpha_j) U(\alpha)$. For the absolute value of $F(z)$ we obtain a bound

$$|F(N + \mathrm{i}\eta + \mathrm{i}\beta)| \leqslant (\mu^2)^N g(N, \{p\}) \tag{27}$$

with

$$g(N, \{p\}) = \int_0^\infty \prod_{i=1}^{I} \mathrm{d}\alpha_i \, U^{-D/2} \mathrm{e}^{-R/U} (W'/U)^{N+\eta} \tag{28}$$

where $\{p\}$ denotes the external momenta.

Taking I-dimensional spherical coordinates, the radial integration may be performed explicitly, taking into account the homogeneity properties of the polynomials R, W' and U in the α variables ($R(\alpha)$ and $W'(\alpha)$ are homogeneous of degree $L + 1$ and $U(\alpha)$ is homogeneous of degree L). We obtain an expression in terms of an integral over the I-dimensional angular variables Ω

$$g(N, \{p\}) = \Gamma(I + N + \eta - \tfrac{1}{2}DL + 1) \int \mathrm{d}\Omega \, f(\Omega)[g(\Omega)]^{N+\eta} \tag{29}$$

where $f(\Omega)$ and $g(\Omega)$ are regular functions, and since the above integral in over angular variables, it has an upper bound, where $K_N(\{p\})$, K_N is a positive quantity. From equations (25), (27) and (29) we see that

$$|R_N(\lambda)| < \frac{\Gamma(N + \eta + I - \frac{1}{2}DL + 1)}{\Gamma(N + 1)} \frac{A}{\eta(1 - \eta)} (\mu^2)^{N+\eta} K_N(\{p\}) \lambda^{N+\eta} \tag{30}$$

where A is a constant independent of N and η.

For $I - \frac{1}{2}DL > 0$ (which is just the condition for UV convergence for the graph), the ratio

$$\kappa = \frac{\Gamma(N + \eta + I - (DL/2) + 1)}{\Gamma(N + 1)}$$

is clearly a finite positive quantity. Thus renaming the various constants appearing in the expressions above, the remainder of the asymptotic expansion may be written in the form

$$|R_N(\lambda)| < K_1 K_N(\{p\})(\mu^2)^N \lambda^N. \tag{31}$$

The scaling parameter λ is arbitrarily small in the limit of the masses going to zero. Therefore, the factor $(\lambda\mu^2)^N$ in the bound above makes the sequence of the remainders $R_N(\lambda)$ converge to zero as $N \to \infty$ which is a condition for convergence of the asymptotic expansion.

4. Concluding remarks

The study of the asymptotic behaviour of Feynman amplitudes has a long history. Weinberg [25] proved such a theorem on asymptotic behaviours for the specific case of scaling by λ all external momenta of a Euclidean convergent amplitude. Later the theorem was extended to divergent amplitudes [24, 26]. In applications of field theory to critical phenomena, the examples of models of field theory that have been found to give relevant information, are controlled by the free field fixed point, or by fixed points that approach the free field fixed point in some limit. This means that the Feynman diagram approach to field theory plays an important role in understanding physical situations in critical phenomena. As we have stressed in the introduction, particularly important are the large-distance correlations in statistical systems, which play a crucial role in the study of phase transitions. In field theory language, these large-distance correlations manifest themselves as infrared divergences of Feynman amplitudes, which are a particular case of the asymptotic behaviour of Feynman amplitudes as we have presented in this work. This is one of the reasons why the analysis presented in this paper could be interesting to the perturbative field-theoretical approach to critical phenomena.

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